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A New Look at Structural Reliability and Risk Theory

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In any situation where there exist random elements in the load history imposed upon a structure, or in its structural performance or strength decay properties, its survival to any lifetime becomes a matter of chance. Reliability theory relates survivorship or risk of failure to the time variable. For the prediction of future performance a mathematical (statistical) model is devised representing the physical system and the applied loads. Values of the relevant parameters are either derived from a look at past performance or are attributed on the basis of judgement, and the outcome is determined by following the rules. As collapse of a structure is a catastrophe, the service life must be limited such that the risk of collapse is acceptably small. In fail-safe situations where inspection and withdrawal of defectives can pre-empt a failure, survivors of inspection can safely be kept in service until few of the originals remain. The aim in this development of the theory is to determine risk rate and probability of survival as functions of time, so that one may judge when structures should be withdrawn from service on grounds of unreliability.

Nomenclature

$F(t)$, $F(n)$ = unreliability: probability of failure before time t or load cycle n
$F_f(t)$ = probability of failure by fatigue before t
$f(H)$, $f(U_0)$ = probability density of H or U_0
$f(t)$ = probability density of failures at time t
$f(n)$ = fraction of original population failing at (discrete) load cycle n
H; H_i; H_p; H_F = characteristic strength decay time; time to crack initiation; time for crack propagation to ℓ_{cr}; time to strength collapse
k_r = fraction of H at which $\ell = \ell_r$
ℓ; ℓ_{cr}; ℓ_r = crack length; critical crack length; crack length for withdrawal at inspection
$m(U)$ = frequency of exceedance of load U
m_0 = total number of load occurrences of all magnitudes per unit time
n = number of load cycles
$R(t)$; $R_0(t)$ = population reliability with strength decay; or with virgin strength preserved
$R(t|U_0,H)$; $R_0(t|U_0,H)$ = element reliability with strength decay; or with virgin strength preserved
$R_\tau(t)$ = population reliability with inspection at time τ
$\bar{r}(t)$; $\bar{r}_0(t)$ = population instantaneous average risk with strength decay; or with virgin strength preserved
$r_F(t)$ = risk of failure through reaching the point of instantaneous strength collapse
$r(t|U_0,H)$; $r_0(t|U_0,H)$ = element risk at time t with strength decay; or with strength preserved
U; U_0; $\tilde{U}_0$ = strength; virgin strength; and median strength
V = applied load
Π = product
ψ = strength decay function of crack length
ϕ = crack length function of time
ζ = strength decay function of time
ζ_F = fraction of virgin strength reached at H_F
τ = time at an inspection

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Introduction

IN everyday life we commonly distinguish between people who are reliable and people who are not. A reliable person is one who always does what he says he will do. An unreliable person is one about whom there is uncertainty as to whether or not he will do what he says he will do—whether or not he does it is a matter of will, chance or luck. Similarly, a machine which works when it is required to work is classed as reliable, whereas one which sometimes does and sometimes does not, (i.e., which cannot with certainty be expected to work) is classed as unreliable. Thus reliability goes with certainty and unreliability goes with uncertainty.

In contrast are the semantics of the reliability analyst. He is concerned not with determinate processes but with processes that involve chance, or, as our dictionary defines it, "luck." His definition of "reliability" is "the likelihood that a thing will perform its desired function for a desired period (its lifetime)."[1] "Unreliability" is "the probability that it will fail to perform its function during its lifetime." When the thing is a structure, failure to perform its function means failure to survive; thus "reliability" means "probability of survival." The use of reliability analysis does not confer improved survival properties upon a structure.

The "risk" or more properly the "risk rate,"[1] is the probability that, at one single event of its whole life experience or during one elementary time interval dt, the device will fail to perform its function. "Risk rate" and "hazard rate" are synonymous.

In thinking about the reliability or the probability of failure of an article, the reliability analyst does not think about an individual. He conceives of an infinite population of which the individual is a representative member. The proportion of this population surviving to time t becomes the probability of survival of the individual, i.e., its reliability. Similarly, the proportion of these survivors which fails at the next event becomes the probability of failure of the surviving individual at that event, i.e., the risk rate. Where the independent variable is time, and where time is continuous, the risk rate is the (average) frequency of occurrence of failing loads, and therefore of failures, per unit time.

Survivorship or reliability functions, expressed as functions of time of exposure to the risk, have characteristic shapes[2]: some of these are illustrated in Fig. 1. Curve A shows the curve for accident risk; curve B is a typical wearing-out survivorship function; curve C is a typical human survivorship function; and curve D illustrates the effect of acclimatization by which the exposed population gradually develops immunity to the risk.

The aim, in this development of reliability theory, is to be able to determine risk rate and probability of survival as functions of time, so that one can judge when structures should be withdrawn from service on grounds of "unreliability."

Basic Concepts

By definition, the reliability $R(t)$ or $R(n)$ is the probability of survival to time t or the nth event. In this study $F(t) = 1-R(t)$, or similarly $F(n)$ is used for unreliability, viz. the probability of failure in the interval from time zero to time t or to load cycles n.

The risk rate $r(t)$ or $r(n)$ has been defined above as the rate of failure (number failing per total number existing) per unit time at time t or per cycle at cycle n.

R and r are related, as will be shown. Let us consider a population of equal and constant strength U subjected to a sequence of random service loads V (i.e., the magnitude of the loads in the sequence is a random variable); then, there will be a certain probability r that the next load occurring will exceed U and will therefore cause an immediate failure. The probability of survival is $(1-r)$ and the probability of survival of n such events is $(1-r)^n$, so

$$R(n) = (1-r)^n \tag{1}$$

$$\text{and approximately} = \exp(-rn) \tag{2}$$

provided rn is small and n is large compared to unity.

If the risk varies with time, and $r = r(1)$, $r(2)$, $r(3)$...at $n = 1,2,3,\ldots$

$$R(n) = \prod_{i=1}^{n} \{1-r(i)\} \tag{3}$$

where Π signifies the product of terms like $\{1-r(i)\}$.

Now approximately

$$R(n) = \exp\left\{-\sum_{i=1}^{n} r(i)\right\} \tag{4}$$

with the same provisos as for Eq. (2).

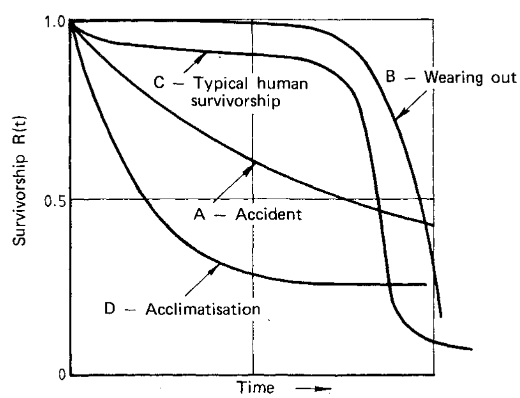

Fig. 1 Typical survivorship functions.

Writing $f(n)$ as the fraction of the original population failing at load cycle n or similarly $f(t)$ as the probability density of failure at time t,

$$f(n) = R(n-1) - R(n)$$

$$= \prod_{i=1}^{n-1} \{1-r(i)\}\cdot r(n) = R(n-1)\cdot r(n) \tag{5}$$

Thus

$$r(n) = f(n)/R(n-1) = f(n)/\{1-F(n-1)\} \tag{6}$$

In the limiting case of large n, and with $n = m_0 t$, t being continuous, the right-hand side of Eq. (6) becomes the derivative of $\ln\{1-f(t)\}$, whence exactly

$$R(t) = \exp\left\{-\int_0^t r(t)\,\mathrm{d}t\right\} \tag{7}$$

and

$$r(t) = f(t)/R(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\{R(t)\}/R(t) \tag{8}$$

It should be noted that, in Eqs. (6) and (8), $f(n)$ is the probability density function of time to failure; $f(t)\,\mathrm{d}t$ is the proportion of the original total which fails in the interval $\mathrm{d}t$; and $r(t)\,\mathrm{d}t$ is the proportion of survivors to time t failing in the interval $\mathrm{d}t$.

It will be appreciated that at early times when the population is not significantly depleted, and when $R(t)$ is sensibly equal to unity, one may approximately equate $r(t)$ and $f(t)$.

The above analysis applies to the case where the risk on all members of the population is the same. Where the population consists of different parts suffering different risks, the derivation of reliability normally requires that the various parts be treated separately and that the total reliability be obtained by summing the reliabilities of the separate parts.

Sources of the Element of Chance in Structural Problems

In aircraft structural problems the chance element arises from two sources, the loads acting on the structure and its strength properties.

Broadly speaking, the sources of more-than-average load on an airplane in flight arise either from intentional pilot action, such as turns, pull-outs, and aerobatic maneuvers, from actions where pilot skill falls short of absolute perfection, e.g., in heavy landings, or from actions of nature such as atmospheric turbulence. The recording of such events over a long period of time gives data on the frequency of exceedance as a function of threshold level; this becomes a smooth curve and is usually and perhaps wrongly called a "spectrum." Such spectra will cover the range from small events which occur many times per hour to extreme events which might recur only once in the lifetime of a large fleet.

Structures, however, are like people—no two have precisely the same strength. Naturally occurring materials such as wood display different strengths in different pieces, and so do manmade aircraft materials. However precise the manufacture, structures display even greater variability. From strength tests on sets of similar structures one can establish a statistical distribution of the virgin strength. Where in any particular case the distribution has not been determined, one can attribute the variability and distribution observed in other structures or in small specimens.

Strength deterioration occurs from fatigue, corrosion, or some other inherent change in the material. In this paper the effects of fatigue, in particular, are discussed. Fatigue comprises the stages of crack initiation and propagation; in the latter stage, strength falls below the virgin strength. In some but not all circumstances the crack size can be measured at regular intervals along the lifetime of a structure, but one cannot determine the strength—even by test—more than once. One can, however, estimate the strength, albeit with some uncertainty, from the crack length. As was noted above, structures are like people—age comes on earlier in some individuals than in others; likewise, the time taken for cracks to reach a given length or for the strength to fall to a given fraction of virgin strength is variable. The variability in time for strength decay or for crack growth can be expressed as a statistical distribution.

Thus variability arises in the differing recurrence frequency of applied loads of different magnitudes, in the differing virgin strengths of different members of the population, and in the differing time taken for their cracks to reach a given size or for their strengths to fall by a nominated amount.

Mathematical-Structural-Statistical Model Representing the Real Structural Situation

To recapitulate, structural reliability theory is concerned with the chance of a catastrophically high load occurring at any time on a structure which has been and is continually exposed to the sequence of loads which are promoting the fatigue process of initiation and growth of cracks and consequently a steady decay in strength. One must therefore construct a mathematical model which embodies the occurrence of potentially catastrophic loads and structural strength decay.

The frequency with which potential failing load thresholds are exceeded is expressed here as a function of the ratio of a chosen strength threshold to the population median virgin strength:

$$m(U/\tilde{U}_0) \tag{9}$$

$m(U/\tilde{U}_0)$ is therefore the risk of failure of a structure whose strength is U. Two such load exceedance curves are illustrated in Fig. 2, one for maneuver loads on military fighter-type aircraft and the other for gust loads on civil transport aircraft.

The decay of strength caused by fatigue is related to crack growth. The relationship of strength to crack size is derived from strength tests of the structures, or of structures regarded as representative, containing fatigue cracks of a variety of sizes.

The relationship is expressed in this paper in functional form:

$$U = U_0\psi(\ell/\ell_{cr}) \tag{10}$$

where ℓ_{cr} is the crack length at which the strength of the average structure is reduced to limit strength, or two-thirds of its virgin strength. This relationship is typically illustrated in the $0yz$ plane of the three-dimensional diagram in Fig. 3, in which crack length is plotted along the y axis and strength U along the z axis. Time is represented along the $0x$ axis.

In some materials strength falls almost linearly with crack length. On the other hand, in a material whose behavior follows the laws of linear elastic fracture mechanics up to general yield, there is no decay of strength until a particular crack length is reached and thereafter it decays as the inverse square root of crack length.

Crack behavior is a function of the number of fatigue load cycles, and, when these occur at a constant rate, is a function of time. Crack behavior comprises an initiation time H_i and a characteristic propagation time H_p. This characteristic time may be defined as the time needed for the crack to propagate completely through the structure (if it could do so without

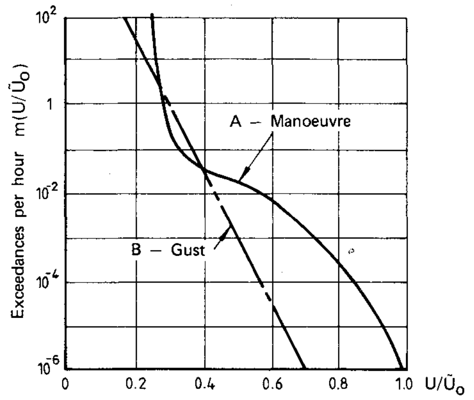

Fig. 2 Typical aircraft gust and maneuver load exceedance spectra.

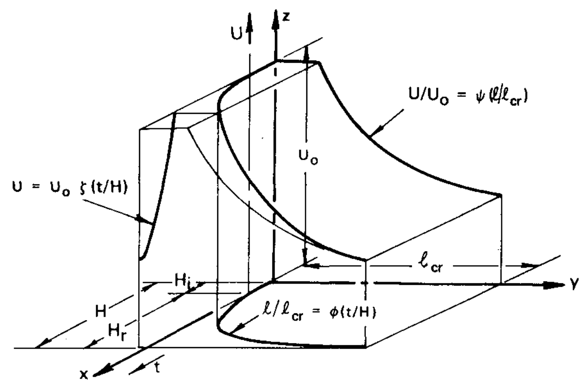

Fig. 3 Crack length-strength-time relationship.

collapse), to the point where the structure has just level-flight strength (if it could do so without a small load cycle causing collapse), or to the critical crack length ℓ_{cr} as defined above. In this study we use the last interpretation.

We write $H = H_i + H_p$ as the individual structure's characteristic time. Thus we write

$$\ell/\ell_{cr} = \phi\{(t-H_i)/H_p\} \tag{11}$$

In some studies[3,4] (including this one) H_i is always assumed to be a given fraction of H_p, so that one may write

$$\ell/\ell_{cr} = \phi(t/H) \tag{12}$$

which we have illustrated in the $0xy$ plane of Fig. 3. (Other authors have assumed H_p to be constant for all members and H_i to be variable.)

The decay in strength with time is thus

$$U = U_0\psi(\ell/\ell_{cr}) = U_0\psi\{\phi(t-H_i)/H_p\} \tag{13}$$

and in our case

$$U = U_0\psi\{\phi(t/H)\} = U_0\zeta(t/H) \tag{14}$$

This is shown by projecting onto the $0xz$ plane of Fig. 3. This figure indicates a continuously increasing crack length and a consequential continuously decreasing strength with time. If, as is sometimes done,[4] these functions are assumed to reach infinite rates, it becomes important (solely for the calculation of risk) to define the time $H_F = k_F H$, and the fraction ζ_F of the virgin strength, at which the instantaneous loss of strength occurs.

Completion of the definition of the model requires the nomination of the parameters involving variability. In some analyses[3] variability is attributed only to H, in others to U_0 and H_i, while in yet others[4,5] as in this study, variability is attributed to U_0 and H while making H_i a constant fraction of H.

If a structure is inspectable, there will be a crack length ℓ_r that is regarded as the limiting permissible crack size. We will assume that this crack length is reached at a time $k_r H$. If an inspection is made at time τ, then all structures with $k_r H < \tau$ will have cracks causing withdrawal and all those with $k_r H > \tau$ will not. Thus the population remaining to just before τ will be divided into two parts, according to the value of H: those with $H < \tau/k_r$ will be eliminated by inspection and those with $H > \tau/k_r$ will remain in the ongoing fraction. The survivorships after an inspection are obtained by integration of the variable H over the range $\tau/k_r < H < \infty$.

Safe-Life Situation

If the fatigue-critical areas of the structure are not inspectable, or if it is considered that unsafe cracks could possibly be present and avoid detection, the whole population may be used only so long as its probability of survival does not fall below an acceptable level. The time at which this limit is reached is the "safe life." As the acceptable probability of survival may be arrived at by the exercise of corporate judgment, it is convenient to evaluate the survivorship over a range of lifetimes.

If the population has no strength variability (either in virgin strength or decay time), then its strength decays according to $U = \tilde{U}_0 \zeta(t/H)$, so that the risk is

$$r(t) = m(U/\tilde{U}_0) = m\{\zeta(t/H)\} \tag{15}$$

and the reliability at t is, from Eq. (7),

$$R(t) = \exp\left[-\int_0^t m\{\zeta(t/H)\}\,\mathrm{d}t\right] \tag{16}$$

If the population has no variability attributed to it in virgin strength but only in strength decay time, then the distribution has a density function denoted $f(H)$, so the reliability must be determined for each element $f(H)\ \mathrm{d}H$ and summed over the distribution:

$$R(t) = \int_{H=0}^{\infty} \exp\left[-\int_0^t m\{\zeta(t/H)\}\,\mathrm{d}t\right] f(H)\,\mathrm{d}H \tag{17}$$

A population assumed to have variability both in virgin strength U_0 and in strength decay time H, is represented by the bivariate distribution with density functions $f(U_0)$ and $f(H)$, so that the reliability is integrated over both of these variables:

$$R(t) = \int_0^\infty \int_0^\infty \exp\left[-\int_0^t m\left\{\frac{U_0\zeta(t/H)}{\tilde{U}_0}\right\}\mathrm{d}t\right] \times f(U_0) f(H)\,\mathrm{d}U_0\mathrm{d}H \tag{18}$$

which may be written as

$$R(t) = \int_0^\infty \int_0^\infty R(t \mid U_0, H) f(U_0) f(H)\,\mathrm{d}U_0\mathrm{d}H \tag{18a}$$

Since $m\{U_0\zeta(t/H)/\tilde{U}_0\}$ is the risk at time t on an element $f(U_0)f(H)\,\mathrm{d}U_0\mathrm{d}H$ of virgin strength U_0 and decay parameter H, and since

$$\exp\left[-\int_0^t m\left\{\frac{U_0\zeta(t/H)}{\tilde{U}_0}\right\}\mathrm{d}t\right]$$

is its survivorship at time t, the instantaneous average risk at time t is

$$r(t) = \left\{\int_0^\infty \int_0^\infty m\left\{\frac{U_0\zeta(t/H)}{\tilde{U}_0}\right\} \times \exp\left[-\int_0^t m\left\{\frac{U_0\zeta(t/H)}{\tilde{U}_0}\right\}\mathrm{d}t\right] \times f(U_0) f(H)\,\mathrm{d}U_0\mathrm{d}H\right\} \Big/ R(t) \tag{19a}$$

or

$$(r)t = \left\{\int_0^\infty \int_0^\infty m\{\quad\} \cdot R(t \mid U_0, H) f(U_0) f(H)\,\mathrm{d}U_0\mathrm{d}H\right\} \Big/ R(t) \tag{19b}$$

It may be noted that any part of the bivariate distribution for which the element survivorship $R(t \mid U_0, H)$ is zero will not survive to time t, nor does it contribute to the average risk rate $r(t)$, even though it falls within the bounds of integration.

The case of a strength decay curve with instantaneous collapse from strength $\zeta_F U_0$ to zero at time $H_F = k_F H$ deserves special consideration. Although not affecting the calculation of survivorship in Eqs. (17) and (18), it does modify the calculation of risk in Eq. (19a). It is perhaps easiest approached by considering the probability distribution of the parameter H_F, where $f(H_F) = f(H)/k_F$. Consider a slice of the bivariate population bounded by $\mathrm{d}H_F$

$$dH_F \int_0^\infty f(U_0)\,\mathrm{d}U_0$$

For any such slice with $H_F < t$, the strength has already fallen to zero and the survivorship $R(t \mid H_F < t)$ is zero, that is, it contributes nothing to either $R(t)$ or $r(t)$. For any slice with $H_F > (t + \mathrm{d}t)$ the survivorship $R(t \mid H_F > t)$ is, in general, finite, and its contribution to the numerator of $r(t)$ is

$$f(H_F)\,\mathrm{d}H_F \int_0^\infty m\left\{\frac{U_0\zeta(t/H_F)}{\tilde{U}_0}\right\} \times \exp\left[-\int_0^t m\left\{\frac{U_0\zeta(t/H_F)}{\tilde{U}_0}\right\}\mathrm{d}t\right] f(U_0)\,\mathrm{d}U_0$$

and this is in fact, when integrated over H_F, the numerator of Eq. (19a.)

Now consider a slice $\mathrm{d}H_F$ with $t < H_F < t + \mathrm{d}t$, i.e., the slice of the population which reaches its instantaneous strength-collapse point in the interval from t to $t + \mathrm{d}t$. Its survivorship at t is [from Eq. (18)]

$$\int_0^\infty \exp\left[-\int_0^{H_F} m\left\{\frac{U_0\zeta(t/H_F)}{\tilde{U}_0}\right\}\mathrm{d}t\right] f(U_0)\,\mathrm{d}U_0$$

and at $t + \mathrm{d}t$ it is zero. The loss from the population in time $\mathrm{d}t$ is

$$f_{H_F}(t)\,\mathrm{d}t \int_0^\infty \exp\left[-\int_0^{H_F} m\left\{\frac{U_0\zeta(t/H_F)}{\tilde{U}_0}\right\}\mathrm{d}t\right] f(U_0)\,\mathrm{d}U_0$$

so that it contributes to the risk rate $r(t)$ an additional

$$r_F(t) = \left\{f_{H_F}(t) \int_0^\infty \exp\left[-\int_0^{H_F} m\left\{\frac{U_0\zeta(t/H_F)}{\tilde{U}_0}\right\}\mathrm{d}t\right] \times f(U_0)\,\mathrm{d}U_0\right\} \Big/ R(t)$$

Expressing the equation in words, this additional risk r_F at time t is the product of the density function $f(H_F)$ and the integrated survivorship over U_0 of the slice $\mathrm{d}H_F$, divided by the total population survivorship. It is the "risk of failure through members reaching the point of instantaneous strength decay," and is independent of the load applied at that instant. This is, in fact, the correct evaluation of the risk sometimes ambiguously described as the "risk of fatigue fracture" or the "risk due to the fatigue crack reaching such an extent that the structure is unable to sustain the mean load."

With a spectrum such as A of Fig. 2, there is a significant risk of failure due to loads which exceed the virgin strength. As structures encountering these loads will fail whether or not they are fatigue cracked, the author subtracts them from the total count of failures and attributes to fatigue the difference between the survivorship with virgin strength preserved and that with strength decay occurring. The risk with strength preserved we call the "risk of ultimate failure," which is conditional on original strength U_0 being preserved. Thus

$$r_0(t) = m(U_0/\bar{U}_0) \tag{20}$$

and for the population with distributed U_0,

$$R_0(t) = \int_0^\infty \exp\{-m(U/\bar{U}_0)\cdot t\} f(U_0)\,\mathrm{d}U_0 \tag{21}$$

Note that the element $f(U_0)\,\mathrm{d}U_0$ has constant $U_0/\bar{U}_0$ for all t, so that $\int_0^t m(U_0/\bar{U}_0)\,\mathrm{d}t$ is equal to $m(U_0/\bar{U}_0)\cdot t$.

Thus the probability of fatigue failure $F_f(t)$ before the time t, i.e., the probability of failure of weakened structures by loads below their virgin strengths, is

$$F_f(t) = R_0(t) - R(t) \tag{22}$$

where the first term comes from Eq. (21) and the second from Eq. (18). Of course in cases such as the civil spectrum, in which the risk of ultimate failure is truly negligible, the first term in Eq. (22) is unity.

The instantaneous risk on an element at time t is given by Eq. (15) or Eq. (20). The population average instantaneous risk at time t can be obtained from the basic Eq. (8). Thus

$$\bar{r}(t) = f(t)/R(t) = \left[\frac{\mathrm{d}}{\mathrm{d}t}(R(t))\right] \Big/ R(t) \tag{23}$$

If $R(t)$ is not significantly different from unity and if the integral in the exponential expression is small, the exponential term in $R(t)$ in the numerator can be approximated as

$$R(t) = \int_0^\infty \int_0^\infty \left[1 - \int_0^t m\left\{\frac{U_0\zeta(t/H)}{\bar{U}_0}\right\}\mathrm{d}t\right] f(U_0) f(H)\,\mathrm{d}U_0\mathrm{d}H$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(R(t)) = \int_0^\infty \int_0^\infty m\left\{\frac{U_0\zeta(t/H)}{\bar{U}_0}\right\} f(U_0) f(H)\,\mathrm{d}U_0\mathrm{d}H$$

thus giving, approximately

$$\bar{r}_e(t) = \int_0^\infty \int_0^\infty m\left\{\frac{U_0\zeta(t/H)}{\bar{U}_0}\right\} f(U_0) f(H)\,\mathrm{d}U_0\mathrm{d}H \tag{24}$$

If this approximation $\bar{r}_e(t)$ is then taken to be the risk suffered uniformally by the whole population, it leads back to an approximate value of reliability:

$$R_e(t) = \exp\left\{-\int_0^t \int_0^\infty \int_0^\infty m\left\{\frac{U_0\zeta(t/H)}{\bar{U}_0}\right\} \times f(U_0) f(H)\,\mathrm{d}U_0\mathrm{d}H\mathrm{d}t\right\} \tag{25}$$

The difference between this approximate equation and the exact Eq. (18) is noted; where $R(t)$ is not significantly different from unity the expressions are not significantly different.

As an example, calculations have been made for a population of structures with rapidly accelerating crack growth and strength decay properties as given by the laws of linear elastic fracture mechanics, while the load spectrum is that for a military fighter-type airplane.

The following data apply:

1) The frequency of loads is given by Curve A of Fig. 2:

$$m(U/\bar{U}_0) = 10^{(A + BU/\bar{U}_0 + CU^2/\bar{U}_0{}^2 \ldots)}$$

where $A = 24.4$, $B = 188.1$, $C = 528.1$, $D = 717.4$, $E = 467.1$, $F = 120.3$.

2) The crack propagation function is

$$\ell/\ell_{cr} = (t/H)^9 - 1.6\times 10^{-6}$$

3) The strength decay function is unity to $t/H = 0.91$; then

$$U/U_0 = 0.67[(t/H)^9 - 1.6\times 10^{-6}]^{-1/2}$$

4) The crack propagation time H to length ℓ_{cr} is log-normally distributed about a median value H of 8000 h with standard deviation (of $\log_{10}H$) of 0.167.

5) Virgin strength U_0 is assumed uniform in one case, while in a second case it is assumed Weibull-distributed with a standard deviation of 0.03, typical of metal structures.

The results of the calculation are illustrated in Fig. 4. In the upper part, survivorship is plotted against time, while in the lower part, risk rate is plotted against time. Results for the population with uniform virgin strength are shown in dotted line. Here, the total reliability $R(t)$ falls steadily to about 2500 h, after which it falls at a more rapid rate, reaching $R(t) = 0.5$ at 8000 h. The plot of $R_0(t)$, the reliability with virgin strength preserved, is continuous with the plot of total reliability $R(t)$ for values of t less than about 2000 h, indicating that for early parts of the lifetime the dominant factor is the probability of failure under loads greater than the virgin strength. This risk is present throughout life, but later the fatigue risk dominates the total. In this latter phase, the reliability at any lifetime is the reliability of the population if no loads occurred greater than limit load. It is therefore the same as would be calculated by conventional safe-life analysis, which is based on the concept that the structure becomes unairworthy when its strength falls to the level at which it would fail in its fatigue test (usually the limit load, in a military airplane).

If a "safe life" were set at that life for a probability of failure of 1 in 1000 (reliability of 0.999), this probability would be reached at 2000 h (one quarter of the median survived life). It is clear, however, that in this example this result would be obtained in the regime where ultimate static failure is dominant. In this example the use of reliability theory, with its automatic inclusion of the static failure risk, produces a rather artificial figure for a safe life. A more logical safe fatigue life is that life at which the total probability exceeds the static failure probability by the acceptable amount.

The results for a population with distributed virgin strength are plotted in full line. Here the reliability function is similar to that for uniform virgin strength, but lies below it for lives up to about 4000 h. In this region, where static failure risk is dominant, the reliability is lower because the larger risk on members with strength below the mean value is not balanced by the smaller risk on members with strength above the mean value. In the lower area are plotted the total risk rate $\bar{r}(t)$ and the risk rate for ultimate static failure $\bar{r}_0(t)$.

The initial risk is approximately 5×10^{-6}, and the static ultimate failure risk falls slightly with time as a result of the "survival of the fittest" effect. The fatigue risk rate only becomes detectable at about 1000 h and grows by about two orders of magnitude over the next 7000 h.

Safe-by-Inspection Situation

Structures may be safe-by-inspection provided that the location of fatigue failure is inspectable, that inspection techniques are available to detect cracks of a size at which the strength is not greatly reduced, and that inspection for the removal of defectives can combine a limiting crack size and an inspection frequency so that an acceptable risk level is not exceeded. To do this, one must be able to calculate the survivorship and the risk rate after an inspection at which members with a crack size larger than the "withdrawal crack size" are withdrawn.

Crack size did not enter explicity into the reliability in the safe-life case; it entered only implicity in the expression $U = U_0\zeta(t/H)$ of Eq. (14). Here we will assume that the crack reaches the rejection crack size in time H_r. This time can be found from Eq. (12) given the rejection size ℓ_r. In our case we define $k_r = H_r/H$. Naturally this crack size must be greater than the smallest crack that can be detected, otherwise safety-by-inspection becomes impracticable.

Thus an inspection at time τ will divide the population into two parts: those with $H<\tau/k_r$ and those with $H>\tau/k_r$. The reliability of the population at some time t after an inspection, given by Eq. (18) above, can be split into two ranges of integration:

$$R(t) = \int_{H=0}^{\tau/k_r}\int_0^{\infty}\exp\ldots dU_0dH + \int_{H=\tau/k_r}^{\infty}\int_0^{\infty}\exp\ldots dU_0dH \quad (26)$$

The first term comprises those not surviving withdrawal at inspection at time τ, while the second term comprises those surviving the inspection and also surviving the risks over the period from zero to t. The latter survivorship (conditional on surviving inspection at time τ) is written $R_\tau(t)$:

$$R_\tau(t) = \int_{\tau/k_r}^{\infty}\int_0^{\infty}\exp\left[-\int_0^t m\left\{\frac{U_0\zeta(t/H)}{\tilde{U}_0}\right\}dt\right]$$
$$\times f(U_0)f(H)\,dU_0dH \quad (27)$$

The average instantaneous risk is similar to $r(t)$ in Eq. (19a), with the limits of integration changed from $0<H<\infty$ to $\tau/k_r<H<\infty$. At early stages in the life, while the population is substantially intact and $R(t)$ is sensibly equal to unity, the average instantaneous risk can be approximated as in Eq. (24). Thus

$$\bar{r}_{e\tau}(t) = \int_{\tau/k_r}^{\infty}\int_0^{\infty} m\left\{\frac{U_0\zeta(t/H)}{\tilde{U}_0}\right\}f(U_0)f(H)\,dU_0dH$$

As an example of the safe-by-inspection case, calculations have been made for a population of structures under the same load spectrum as before. The population has the same distribution of virgin strength as before, but has a linear crack growth with time and linear decay of strength with crack size. The crack-propagation time is assumed to be log-normally distributed about a median value H of 5000 h with the same standard deviation (of $\log_{10}H$) of 0.167. The results are illustrated in Fig. 5. Because the virgin strength distribution is

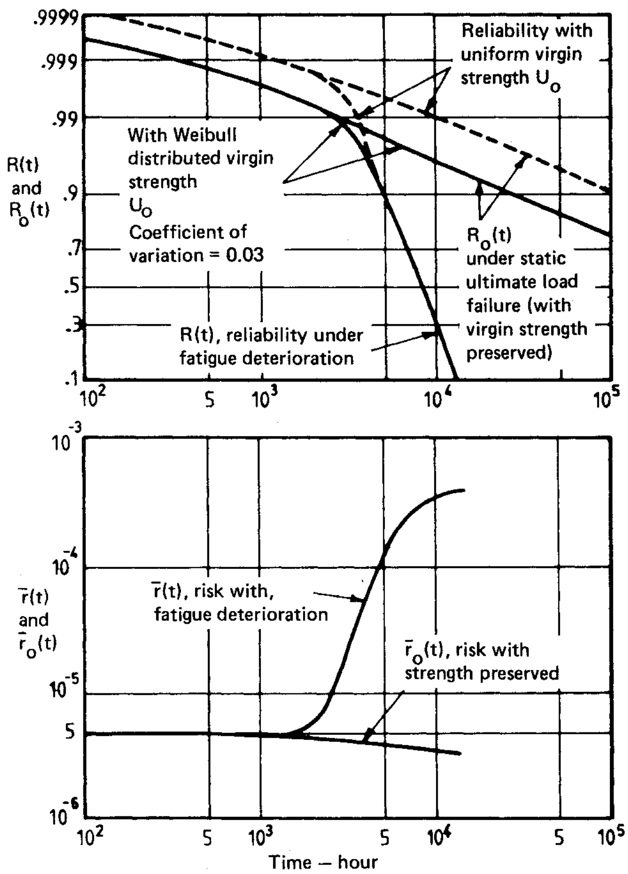

Fig. 4 Survivorship and risk rate with a safe-life philosophy.

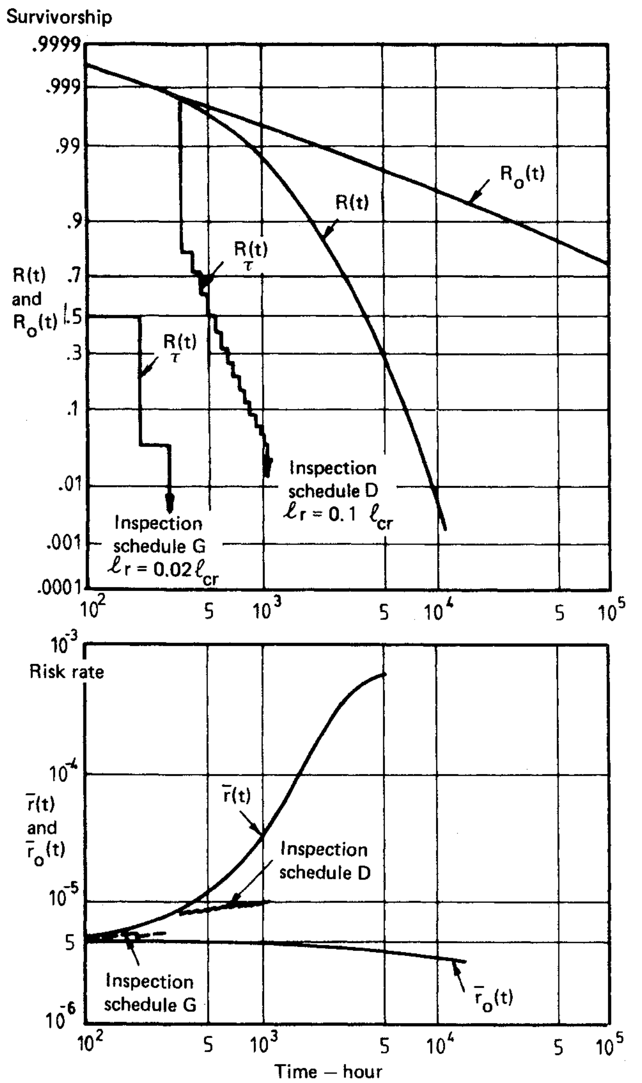

Fig. 5 Survivorship and risk rate with safe-by-inspection philosophy.

as before, the risk of static ultimate failure and the reliability under this risk alone remain as before. However, the combination of linear decay in strength and the shorter characteristic life H causes the survivorship to decline earlier and progressively more than in the previous example, and the risk rate $r(t)$ to escalate earlier.

The effect, at an inspection, of removing structures with a crack size greater than the limit for rejection is to remove from the population those with cracks most advanced, so that the risk suffered by the survivors of inspection will be lowered. This leads to a risk-rate curve generally similar to that without inspections, but translated to the right (i.e., to a later time). The smaller the crack size chosen for rejection, the further to the right lies the risk-rate curve. Thus, the risk on survivors is decreased the smaller the size chosen for rejection, but it follows that the survivors are fewer. The risk rate after an inspection, or averaged between one inspection and the next, or immediately before the next inspection, is a function of rejection crack size and the time inspections are scheduled. If two of these parameters are arbitrarily chosen the third will be determined, as will the survivorship at each inspection.

In this example we have chosen two different schedules, each with regular inspections. The first we denote inspection schedule G (Fig. 5) in which an inspection occurs at every 100 h; the crack size for rejection is chosen as $\ell_r = 0.02\,\ell_{cr}$, at which the strength has fallen to 0.994 of its virgin strength. Examination of the risk rate for schedule G in Fig. 5 shows that the risk rate never grows to more than 6×10^{-6}/h and is never more than 1×10^{-6} greater than the static failure risk rate, but the plot of reliability shows that less than 0.04 of the original population survives the second inspection. The inspection criterion is too severe, for it throws away many structures on which the risk is no greater than on uncracked ones.

Inspection schedule D is arbitrarily chosen with the first inspection at 350 h, and at every 50 h after that. The rejection crack size was set at $0.1\,\ell_{cr}$, five times larger than in the previous case. The risk rate for schedule D, in the lower part of Fig. 5, falls away from the uninspected risk at 350 h, where the risk was 9.0×10^{-6}/h and continues as a sawtooth curve, falling at every 50-h inspection and rising until just before the next. It never exceeds 9.8×10^{-6}, just a little over twice the static failure risk. However, the survivorship function shows that 50% of the population achieves a life in excess of 550 and 20% achieves a life in excess of 700 h.

The total useful life available in the fleet is represented by the area under the survivorship curve. The area under the curve for schedule D is perhaps five times that under the curve for schedule G. Thus, it provides more usage for the fleet. It is substantially less than the area under the curve of $R(t)$, an impractical case where every structure is used until it collapses.

A very important point arises from consideration of the curve $r(t)$ without inspections and that for inspection schedule D. In the former case the risk rises sharply from the time when fatigue failure of short-lived members of the population is reached. As the position of this curve along the life axis is dependent on estimation - sometimes on the result of a single test - there is uncertainty about where the rising slope of the risk curve is located, and this uncertainty is often not covered by a scatter factor.

For the inspectable structure, instanced in the case of inspection schedule D, the risk rate is nearly constant (with small sawtooth oscillations) over a long period of the life. It is obvious from Fig. 5 that the interval between inspections is not very critical; every second one could be left out without greatly enlarging the risk. Since the risk rate is more or less flat, it follows that the risk is not greatly influenced by the population's mean life to failure, so that safety is much less critically dependent upon the numerical values of mean life initially fed into the calculations.

There is, however, a practical case not covered by the above analysis: when a crack is observed in a structure at an inspection, but is not of a size to call for rejection. The structure in question then ceases to be one of a bivariate population distributed in H and U_0 and becomes one in which crack size at time τ is known, so that H is also known. It becmes one of a population distributed in U_0 only, and the reliabiity is

$$R_\tau(t\,|\,H) = \int_0^\infty \exp\left[-\int_0^t m\left\{\frac{U_0\zeta(t/h)}{\bar{U}_0}\right\}\mathrm{d}t\right] f(U_0)\,\mathrm{d}U_0 \qquad (28)$$

The expression for average instantaneous risk rate derived from Eq. (19a) is similarly modified, and the approximate risk rate is

$$r_{e\tau}(t\,|\,H) = \int_0^\infty m\left\{\frac{U_0\zeta(t/H)}{\bar{U}_0}\right\} f(U_0)\,\mathrm{d}U_0$$

Observation of the crack size in such an individual at successive inspections can be used to indicate whether the crack propagation function $\ell/\ell_{cr} = \phi(t/H)$ in practice matches that assumed in the model, or whether the model requires modification.

Acceptable Levels of Risk

The question of what is an acceptable fatigue risk or an acceptable probability of failure in the lifetime still appears to be a matter of arbitrary judgment. The Aeronautical Research Laboratories have for many years used the limiting acceptable figure of a probability of failure of 1 in 1000 in the lifetime for safe-life structures, whether civil or military. For a civil aircraft with a service life of 50,000 h, the corresponding risk rate is 2×10^{-7} failures/h. Lundberg[6] has proposed for civil aircraft that the fatigue accident rate should be no more than 10^{-9} failures/h so by Lundberg's standards the A.R.L. figure is too high when applied to civil aircraft. Lundberg has proposed that an accident rate from all causes of 3×10^{-7} should be aimed at. The statistics for civil aircraft appear to indicate that these figures are being exceeded.

For military fighter-type aircraft, failure rates are much higher. Pugsley[7] has quoted 3.2×10^{-5} as "a little above the average structural accident rate for modern fighter airplanes." Statistics since that data appear to confirm figures of that order. However, the target of 2×10^{-7} referred to above is 1/160 of this structural accident rate. There is, however, always a penalty for excessive safety, in that parts are rejected excessively early. It is the author's view that a figure of 2×10^{-7} is too low a target for fatigue risk in fighter-type airplanes, and that an average value over the lifetime of 2×10^{-6} is a more logical figure. On the other hand, a fatigue risk of the order of 2×10^{-7}/h seems not inappropriate for military transport aircraft, and Lundberg's accident rate goal of 10^{-9}/h is still regarded as desirable for civil aircraft carrying fare-paying passengers.

General Discussion

Applying the principles of reliability theory permits us to determine survivorship and risk rate, with mathematical precision, for the case of populations of structures with distributed and time-varying strength properties when the precise collapse time is random because of the randomness of occurrence of applied loads in sequence.

Reliability analysis is conducted by first constructing a mathematical model of the real population, incorporating the statistical variabilities of load and strength.

The precise determination of the parameters in the model is difficult, because the strength properties of any structure can only be measured once and because the parameters of statistical distributions can only be determined with precision by testing large numbers of population members. A few of the problems are the following: estimating the mean virgin strength from a single test; estimating the mean characteristic life value H from a single test; estimating the coefficient of variation of any of the parameters from a small or nonrepresentative sample; and determining the upper end of a load history curve at frequencies less than once in the lifetime of an airplane or perhaps less than once in the life of a fleet.

Reliability analysis highlights the significance of fatigue risk as compared with ultimate static failure risk, and shows that structures initially ultimate-load-failure sensitive will eventually become fatigue-failure sensitive.

Reliability analysis provides a logical rationale for choosing inspection procedures for safe-by-inspection situations, enabling one to compromise between the factors of fleet survival, permissible risk, crack size for rejection, and intervals between inspections. Because of the effect of inspections in levelling out the risk rate, it is apparent that safety-by-inspection procedures accommodate much more readily to uncertainties in the input data than do safe-life procedures. At the same time, because safe-by-inspection structures are employed until failures in individual structures are imminent, or at least in train, they offer possibilities of verifying (or refuting) prediction of failure rates, and therefore offer possibilities of improving analytical methods with feedback from the real situation.

Reliability means survival and risk means a matter of chance. Analysis can often not change the chances, but it does provide a satisfying philosophy to replace purely arbitrary judgment in establishing safe fatigue lives, safe inspection times and withdrawal crack sizes, for safe-life or fail-safe aircraft structures. The technique is capable of further extension to other transport vehicles and to civil engineering structures subject to similar random characteristics of load and strength.

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